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A supersymmetric extension of infinite dimensional Lie algebras

by

Kimio UENO *) (上野喜三雄)

and

Hirofumi YAMADA **) (山田裕史)

*) Department of Mathematics, Yokohama City University, Yokohama 236, Japan

**) Department of Mathematics, Hiroshima University, Hiroshima 730, Japan

0. Introduction

The transformation theory of the KP (Kadomtsev-Petviashvili) hierarchy was completed in the beautiful work of Date, Jimbo, Kashiwara and Miwa (see e.g., [2]). They have constructed the Fock representation of the Lie algebra $\mathfrak{gl}(\infty)$ and , using the Bose-Fermi correspondence, realized the irreducible components on the polynomial space of infinitely many variables. The Lie algebra $\mathfrak{gl}(\infty)$ has many subalgebras, corresponding to various type of solutions of the KP hierarchy, for example, the Kac-Moody (affine) algebra of type $A_{\ell}^{(1)}$ and the Virasoro algebra.

Our main object is the "supersymmetric extension" of above theory. As the first step we introduce in this note the Lie superalgebra $\mathfrak{gl}(\infty|\infty)$ by making use of the free field operators. We show that the "super Kac-Moody algebras" and the "super Virasoro algebra" are contained as the subsuperalgebra of $\mathfrak{gl}(\infty|\infty)$.

Recently the authors got a preprint of Manin and Radul [5]. They have introduced a supersymmetric extension of the KP hierarchy. We shall discuss the relationship between their theory and $\mathfrak{gl}(\infty|\infty)$ in the subsequent paper.

1. Lie superalgebras

We first define the notion of Lie superalgebras. A $\mathbb{Z}_2$-graded complex vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is called a Lie superalgebra if there is a bilinear bracket product $[\ ,\ ]$ on $\mathfrak{g}$ satisfying the following conditions. If $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$ (α, β = 0,1), then 1) $[x,y] \in \mathfrak{g}_{\alpha+\beta (\mathrm{mod}\ 2)}$, 2) $[x,y] = -(-)^{\alpha\beta}[y,x]$ and 3) $[x,[y,z]] = [[x,y],z] + (-)^{\alpha\beta}[y,[x,z]]$. The last relation is referred to as "super Jacobi identity". The space $\mathfrak{g}_0$ (resp. $\mathfrak{g}_1$) is called the even (resp. odd) part. Remark that a Lie superalgebra is not a Lie algebra.

The simplest example of the Lie superalgebras is constructed by the following manner. Let N = m + n be a positive integer, and let

$$\mathfrak{g}_0 = \left\{ \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} ;\ A \text{ is } m\times m,\ D \text{ is } n\times n \right\},$$
$$\mathfrak{g}_1 = \left\{ \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} ;\ B \text{ is } m\times n,\ C \text{ is } n\times m \right\} \text{ so that } \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

is the space of all N×N comlex matrices. For $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_\beta$ we define $[X,Y] = XY - (-)^{\alpha\beta} YX$. Then the space $\mathfrak{g}$ is a Lie superalgbra which is denoted by $\mathfrak{gl}(m|n)$. For $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{g}$ we define the "supertrace" by str X = tr A - tr D. Supertraceless elements of $\mathfrak{gl}(m|n)$ make a Lie subsuperalgebra $\mathfrak{sl}(m|n)$.

See [1] and [3] for other concepts and examples of Lie superalgebras.

2. The Lie superalgebra $\mathfrak{gl}(\infty|\infty)$

We consider the vector space $V = \mathbb{C}[t,t^{-1},\xi]/(\xi^2)$ with basis $e_j^{(0)} = t^{-j}$, $e_j^{(1)} = t^{-j}\xi$ $(j \in \mathbb{Z})$. Denote V_0 (resp. V_1) the space spanned by $e_j^{(0)}$'s (resp. $e_j^{(1)}$'s). Let $E_{ij}^{(\alpha\beta)}$ $(\alpha, \beta = 0,1;\ i,j \in \mathbb{Z})$ be the endomorphism on V such that $E_{ij}^{(\alpha\beta)} e_k^{(\gamma)} = \delta^{\beta\gamma}\delta_{jk} e_j^{(\alpha)}$. If we define the bracket product for $E_{ij}^{(\alpha\beta)}$'s by

$$[E_{ij}^{(\alpha\beta)}, E_{i'j'}^{(\alpha'\beta')}] = \delta^{\beta\alpha'}\delta_{ji'}E_{ij'}^{(\alpha\beta')} - (-)^{(1-\delta^{\alpha\beta})(1-\delta^{\alpha'\beta'})}\delta^{\beta'\alpha}\delta_{j'i}E_{i'j}^{(\alpha'\beta)},$$

then the space $\mathfrak{gl}(2\infty) = \left\{ \sum a_{ij}^{(\alpha\beta)} E_{ij}^{(\alpha\beta)}\ ;\ a_{ij}^{(\alpha\beta)} = 0 \text{ if } |i-j| \gg 1 \right\}$ has the structure of Lie superalgebra. The even (resp. odd) part is the space of linear combinations of $E_{ij}^{(\alpha\beta)}$'s with $\alpha = \beta$ (resp. $\alpha \neq \beta$).

We can construct the Lie superalgebra $\mathfrak{gl}(2\infty)$ by making use of the "free field operators". Let A be the Clifford algebra over $\mathbb{C}$ with generators $\psi_j^{(\alpha)}, \psi_j^{(\alpha)*}$ $(\alpha = 0,1;\ i,j \in \mathbb{Z})$, satisfying the defining relations:

$$[\psi_i^{(0)},\psi_j^{(0)}]_+ = [\psi_i^{(0)*},\psi_j^{(0)*}]_+ = 0,\ [\psi_i^{(0)},\psi_j^{(0)*}]_+ = \delta_{ij},$$

$$[\psi_i^{(1)},\psi_j^{(1)}] = [\psi_i^{(1)*},\psi_j^{(1)*}] = 0,\ [\psi_i^{(1)},\psi_j^{(1)*}] = -\delta_{ij},$$

$$[\psi_i^{(0)},\psi_j^{(1)}] = [\psi_i^{(0)},\psi_j^{(1)*}] = [\psi_i^{(0)*},\psi_j^{(1)}] = [\psi_i^{(0)*},\psi_j^{(1)*}] = 0.$$

An element of $W^{(0)} = (\bigoplus_{j\in\mathbb{Z}} \mathbb{C}\psi_j^{(0)}) \oplus (\bigoplus_{j\in\mathbf{Z}} \mathbb{C}\psi_j^{(0)*})$ (resp. $W^{(1)} = (\bigoplus_{j\in\mathbb{Z}} \mathbb{C}\psi_j^{(1)}) \oplus (\bigoplus_{j\in\mathbb{Z}} \mathbb{C}\psi_j^{(1)*})$) is referred to as a free fermion (resp. free boson). The following proposition is easy to see.

Proposition 1. The application $E_{ij}^{(\alpha\beta)} \longmapsto \psi_i^{(\alpha)}\psi_j^{(\beta)*}$ defines a representation of the Lie superalgebra $\mathfrak{gl}(2\infty)$.

Next we define the central extension of $\mathfrak{gl}(2\infty)$. First we define the "vacuum expectation value" for quadratic elements in A. Set the linear form by

$$\langle \psi_i^{(\alpha)}\psi_j^{(\beta)}\rangle = \langle \psi_i^{(\alpha)*}\psi_j^{(\beta)*}\rangle = 0,$$

$$\langle \psi_i^{(0)}\psi_j^{(0)*}\rangle = \begin{cases} 1 & i = j < 0 \\ 0 & \text{otherwise}, \end{cases} \qquad \langle \psi_j^{(0)*}\psi_i^{(0)}\rangle = \begin{cases} 1 & i = j \geqslant 0 \\ 0 & \text{otherwise} \end{cases},$$

$$\langle \psi_i^{(1)}\psi_j^{(1)*}\rangle = \begin{cases} -1 & i = j < 0 \\ 0 & \text{otherwise}, \end{cases} \qquad \langle \psi_j^{(1)*}\psi_i^{(1)}\rangle = \begin{cases} 1 & i = j \geqslant 0 \\ 0 & \text{otherwise} \end{cases},$$

and the normalization condition $\langle 1 \rangle = 1$. We put
$:\psi_i^{(\alpha)}\psi_j^{(\beta)*}: = \psi_i^{(\alpha)}\psi_j^{(\beta)*} - \langle \psi_i^{(\alpha)}\psi_j^{(\beta)*} \rangle$, the normal product.

Proposition 2. If we put $Z_{ij}^{(\alpha\beta)} = :\psi_i^{(\alpha)}\psi_j^{(\beta)*}:$, then the following commutation and anti-commutation relations hold:

1) $[Z_{ij}^{(00)}, Z_{i'j'}^{(00)}] = \delta_{ji'}Z_{ij'}^{(00)} - \delta_{j'i}Z_{i'j}^{(00)} + \delta_{ji'}\delta_{j'i}(Y_+(j)-Y_+(i))$,

2) $[Z_{ij}^{(11)}, Z_{i'j'}^{(11)}] = \delta_{ji'}Z_{ij'}^{(11)} - \delta_{j'i}Z_{i'j}^{(11)} - \delta_{ji'}\delta_{j'i}(Y_+(j)-Y_+(i))$,

3) $[Z_{ij}^{(00)}, Z_{i'j'}^{(11)}] = 0$,

4) $[Z_{ij}^{(00)}, Z_{i'j'}^{(01)}] = \delta_{ji'}Z_{ij'}^{(01)}$,

5) $[Z_{ij}^{(00)}, Z_{i'j'}^{(10)}] = -\delta_{j'i}Z_{i'j}^{(10)}$,

6) $[Z_{ij}^{(11)}, Z_{i'j'}^{(01)}] = -\delta_{j'i}Z_{i'j}^{(01)}$,

7) $[Z_{ij}^{(11)}, Z_{i'j'}^{(10)}] = \delta_{ji'}Z_{ij'}^{(10)}$,

8) $[Z_{ij}^{(01)}, Z_{i'j'}^{(01)}]_+ = 0$,

9) $[Z_{ij}^{(10)}, Z_{i'j'}^{(10)}]_+ = 0$,

10) $[Z_{ij}^{(01)}, Z_{i'j'}^{(10)}]_+ = \delta_{ji'}Z_{ij'}^{(00)} + \delta_{j'i}Z_{i'j}^{(11)} + \delta_{ji'}\delta_{j'i}(Y_+(j)-Y_+(i))$,

where $Y_+(j) = \begin{cases} 1 & j \geqslant 0 \\ 0 & j < 0. \end{cases}$

By Proposition 2 the space

$$\left\{ \sum a_{ij}^{(\alpha\beta)} Z_{ij}^{(\alpha\beta)} \quad ; \quad a_{ij}^{(\alpha\beta)}=0 \text{ for } |i-j| \gg 1 \right\} \oplus \mathbb{C}\cdot 1$$

has the Lie superalgebra structure which is the one dimensional central extension of $\mathfrak{gl}(2\infty)$. We denote this Lie superalgebra by $\mathfrak{gl}(\infty|\infty)$.

3. Subalgebras

In this section we give some Lie subsuperalgebras of $\mathfrak{gl}(\infty|\infty)$.

The Lie algebra $\mathfrak{gl}(\infty)$ in the sense of [2] is, of course, a subalgebra of the even part of $\mathfrak{gl}(\infty|\infty)$. In fact, the space

$$\left\{ \sum a_{ij}^{(00)} Z_{ij}^{(00)} \quad ; \quad a_{ij}^{(00)}=0 \text{ for } |i-j| \gg 1 \right\} \oplus \mathbb{C}\cdot 1 \quad \text{is}$$

isomorphic to $\mathfrak{gl}(\infty)$. Define the elements $L_m^{(0)} = -\sum_{j\in\mathbb{Z}} j Z_{j+m\, j}^{(00)}$ for $m \in \mathbb{Z}$. Then we have the commutation relation

$$[L_m^{(0)}, L_n^{(0)}] = (m-n)L_{m+n}^{(0)} + \frac{1}{6}(m^3 - m)\,\delta_{m+n\, 0}.$$

Hence $\bigoplus_{m\in\mathbb{Z}} \mathbb{C}L_m^{(0)} \oplus \mathbb{C}\cdot 1$ is a Lie subalgebra of $\mathfrak{gl}(\infty)$. This algebra is called the "Virasoro algebra" [4].

There are two manners for the supersymmetric extension of the Virasoro algebra, namely the "Ramond algebra" and the "Neveu-Schwartz algebra" [4]. The Ramond (resp. Neveu-Schwartz) algebra is the complex Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where the even part $\mathfrak{g}_0$ has the basis $\{ l_m, c; m \in \mathbb{Z} \}$, and the odd part $\mathfrak{g}_1$ has the basis $\{ g_k; k \in \mathbb{Z} \}$ (resp. $\{ g_k; k \in \mathbb{Z}+1/2 \}$) satisfying the following bracket relations:

$$[l_m, l_n] = (m - n) l_{m+n} + \frac{1}{8}(m^3 - m)\, \delta_{m+n\ 0}\, c,$$

$$[l_m, g_k] = (\tfrac{1}{2}m - k) g_{m+k},$$

$$[g_j, g_k] = 2 l_{j+k} + \frac{1}{2}(j^2 - \tfrac{1}{4})\, \delta_{j+k\ 0}\, c \qquad \text{and}$$

$$[\mathfrak{g}, c] = \{0\}.$$

Proposition 3. Define the elements

$$L_m = -\sum_{j \in \mathbb{Z}} j (Z^{(00)}_{m+j\ j} + Z^{(11)}_{m+j\ j}) - \frac{m}{2} \sum_{j \in \mathbb{Z}} Z^{(11)}_{m+j\ j} + \frac{1}{8} \delta_{m\ 0} \qquad \text{and}$$

$$G_m = -\sqrt{-1} \sum_{j \in \mathbb{Z}} (Z^{(01)}_{m+j\ j} + j Z^{(10)}_{m+j\ j}) \qquad \text{for } m \in \mathbb{Z}.$$

Then $l_m \longmapsto L_m$, $g_m \longmapsto G_m$, $c \longmapsto 2$ is a representation of the Ramond algebra.

Proposition 4. Define the elements

$$L_m = -\sum_{j\in Z} j(Z^{(00)}_{m+j\ j} + Z^{(11)}_{m+j\ j}) - \frac{m-1}{2}\sum_{j\in \mathbb{Z}} Z^{(11)}_{m+j\ j} + \frac{1}{2}\delta_{m\ 0} \quad \text{and}$$

$$G_{m+1/2} = -\sqrt{-1}\sum_{j\in \mathbb{Z}} (Z^{(01)}_{m+j\ j} + jZ^{(10)}_{m+j+1\ j}) \qquad \text{for } m\in \mathbb{Z}.$$

Then $l_m \longmapsto L_m$, $g_{m+1/2} \longmapsto G_{m+1/2}$, $c \longmapsto 2$ is a representation of the Neveu-Schwartz algebra.

In both cases a slight modification gives the more general representation, in which the central element c corresponds to arbitrarily given complex number.

Elements of $\mathfrak{gl}(\infty|\infty)$ are written as $\sum a^{(\alpha\beta)}_{ij} Z^{(\alpha\beta)}_{ij} + a$. Consider the following conditions for the coefficients $a^{(\alpha\beta)}_{ij}$:

1) $a^{(\alpha\beta)}_{i+m^{(\alpha)}\ j+m^{(\beta)}} = a^{(\alpha\beta)}_{ij}$,

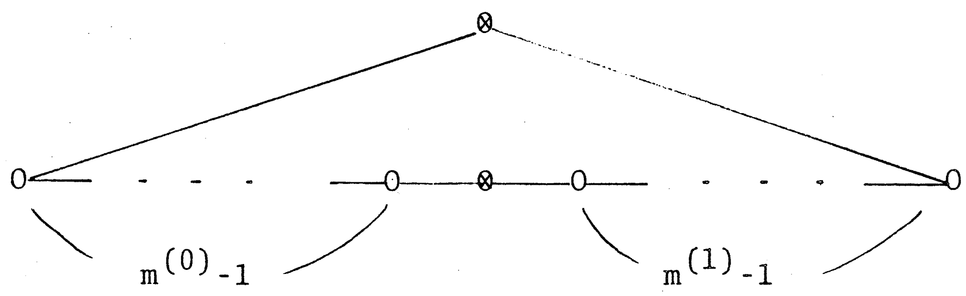
2) $\displaystyle\sum_{i=0}^{m^{(0)}-1} a^{(00)}_{i\ i+jm^{(0)}} - \sum_{i=0}^{m^{(1)}-1} a^{(11)}_{i\ i+jm^{(1)}} = 0$ for any $j\in \mathbb{Z}$.

For the sake of simplicity we assume that $m^{(0)} \neq m^{(1)}$.

Proposition 5. The linear combinations with coefficients satisfying 1) and 2) construct a Lie subsuperalgebra of $\mathfrak{gl}(\infty|\infty)$ isomorphic to the contragredient Lie superalgebra whose Cartan matrix is

$$\begin{bmatrix}
0 & \overset{1}{\cancel{0}} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & -1 \\
-1 & 2 & -1 & & & & & & & \\
0 & -1 & 2 & -1 & & & & \text{\Large 0} & & \\
\vdots & & & \ddots & & & & & & \\
 & & & -1 & 2 & -1 & & & & \\
\vdots & & & & -1 & 0 & 1 & & & \\
 & & & & & -1 & 2 & -1 & & \\
\vdots & & \text{\Large 0} & & & & & \ddots & & \\
0 & & & & & & & -1 & 2 & -1 \\
-1 & & & & & & & & -1 & 2
\end{bmatrix}$$

The corresponding Dynkin diagram is



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